AP Calculus – AB - Problem of the Week
DUE NO LATER THAN 12/03/09 AT 8:00 AM

Name ________________________________
Period _____

Solutions to 11/19/09 POW

1. Let f be the function given by $f(x) = 2xe^{2x}$
   a. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

      $\lim_{x\to\infty} f(x) = \infty$ or DNE $\qquad \lim_{x\to-\infty} f(x) = 0$

      +1 0 as $x \to -\infty$, +1 ∞ or DNE as $x \to \infty$

   b. Find the absolute minimum value of f. Justify your answer is an absolute minimum.

      $f'(x) = 2e^{2x} + (2x)(2)(e^{2x})$

      $0 = 2e^{2x}(1+2x)$

      $x = \frac{-1}{2}$ — +1 solves $f'(x) = 0$

      $f\left(\frac{-1}{2}\right) = \frac{-1}{e} \approx -0.368$ or -0.367 — +1 evaluates $f\left(\frac{-1}{2}\right)$

      $f'(x) < 0$ for all $x < \frac{-1}{2}$ and $f'(x) > 0$ for all $x > \frac{-1}{2}$ — +1 justifies absolute min value

      0/1 for local argument
      0/1 without explicit symbolic derivative
      0/3 if no abs min based on derivative

   c. What is the range of f?

      $\left[\frac{-1}{e}, \infty\right)$ or $[-0.367, \infty)$ or $[-0.368, \infty)$ — +1 answer

      Note: must include left-hand endpt and exclude right hand 'endpoint'

   d. Consider the family of functions defined by $y = bxe^{bx}$, where b is z nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

$y' = be^{bx} + b^2xe^{bx} = be^{bx}(1+bx) = 0$ — +1 sets $y' = be^{bx}(1+bx) = 0$

$x = \frac{-1}{b}$ — +1 solves $y' = 0$

At $x = \frac{-1}{b}$, $y = \frac{-1}{e}$ — +1 evaluates y at critical number

Note : 0 / 3 if only considering specific values of b

2. Suppose that the function f has a continuous second derivative for all x, and that $f(0)=2$, $f'(0)=-3$, and $f''(0)=0$. Let g be a function whose derivative is given by $g'(x)=e^{-2x}(3f(x)+2f'(x))$ for all x.

a. Write an equation of the line tangent to the graph of f at the point where x = 0.

$y-2=-3(x-0)$ +1 equation

b. Is there sufficient information to determine whether or not the graph of f has a point of inflection when x = 0? Explain your answer.

No. Whether $f''(x)$ changes sign at x = 0 is unkown. +1 answer, +1 explanation

The only given value of $f''(x)$ is $f''(0)=0$

c. Given that g(0) = 4, write an equation of the line tangent to the graph of g at the point where x = 0.

$g'(x)=e^{-2x}(3f(x)+2f'(x)) \to g'(0)=0$ +1 $g'(0)$

$y-4=0(x-0)$ +1 equation

$y=4$

d. Show that $g''(x)=e^{-2x}(-6f(x)-f'(x)+2f''(x))$. Does g have a local maximum at x = 0? Justify your answer.

$g'(x)=e^{-2x}(3f(x)+2f'(x))$

$g''(x)=(-2e^{-2x})(3f(x)+2f'(x))+e^{-2x}(3f'(x)+2f''(x))$

$=e^{-2x}(-6f(x)-f'(x)+2f''(x))$ +1 verify derivative

$g''(0)=e^{0}[(-6)(2)-(-3)+2(0)]=-9$ +1 $g'(0)=0$ and $g''(0)=0$

Since $g'(0)=0$ and $g''(0)<0$, g does have a local max at x = 0 +1 answer and reasoning

3. Let f be the function defined by $f(x)=k\sqrt{\ }-\ln x$ for x > 0, where k is a positive constant.

a. Find $f'(x)$ and $f''(x)$.

$f'(x)=\dfrac{k}{2\sqrt{x}}-\dfrac{1}{x}$ +1 $f'(x)$

$f''(x)=\dfrac{-1}{4}kx^{-3/2}+\dfrac{1}{x^2}$ +1 $f''(x)$

b. For what value of the constant k does f have a critical point at x = 1? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at x = 1. Justify your answer.

$f'(1)=\dfrac{1}{2}k-1=0 \to k=2$ +1 sets $f'(1)=0$ or $f'(x)=0$

When $k=2$, $f'(1)=0$ and $f''(1)=\dfrac{-1}{2}+1>0$ +1 solves for k

f has a relative minimum value at x = 1 by the +1 answer

sec ond derivative test. +1 justification

c. For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

At this inflection point, $f''(x)=0$ and $f(x)=0$ +1 $f''(x)=0$ or $f(x)=0$

$f''(x)=0 \to \dfrac{-k}{4x^{3/2}}+\dfrac{1}{x^2}=0 \to k=\dfrac{4}{\sqrt{x}}$ +1 equation in one variable

$f(x)=0 \to k\sqrt{x}-\ln x=0 \to k=\dfrac{\ln x}{\sqrt{x}}$ +1 answer

Therefore, $\dfrac{4}{\sqrt{x}}=\dfrac{\ln x}{\sqrt{x}} \to 4=\ln x \to x=e^4 \to k=\dfrac{4}{e^2}$

POW DUE 12/03/09

1. **(no calculator)** A particle moves along the x-axis with position at time t given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.
   (a) Find the time t at which the particle is furthest to the left. Justify your answer.
   (b) Find the value of the constant A for which x(t) satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

2. **(calculator required)** The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table below gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

x	f(x)	$f'(x)$	g(x)	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

   (a) Explain why there must be a value r for 1 < r < 3 such that h(r) = –5.
   (b) Explain why there must be a value c for 1 < c < 3 such that $h'(c) = -5$.
   (c) Let w be the function given by $w(t) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.
   (d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at x = 2.

3. **(no calculator)** Let f be a twice-differentiable function such that f(2) = 5 and f(5) = 2. Let g be the function given by $g(x) = f(f(x))$.
   (a) Explain why there must be a value c for 2 < c < 5 such that $f'(c) = -1$.
   (b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for 2 < k < 5 such that $g''(k) = 0$.
   (c) Show that if $f''(x) = 0$ for all x, then the graph of g does not have a point of inflection.
   (d) Let $h(x) = f(x) - x$. Explain why there must be a value r for 2 < r < 5 such that $h(r) = 0$