APCBC Problem of the Week　　　　　　　　　　　　　　Name ______________________________

DUE NO LATER THAN 4/1/10 at 8:00 AM

SOLUTIONS TO 3/26/10 POW

1. ***The use of a calculator is NOT PERMITTED on this problem.*** Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let P(x) be the third-degree Taylor polynomial for f about x = 0.

a. Find P(x)

$f(0) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2},\ f'(0) = 5\cos\frac{\pi}{4} = \frac{5\sqrt{2}}{2},\ f''(0) = -25\sin\frac{\pi}{4} = \frac{-25\sqrt{2}}{2},\ f'''(0) = -125\cos\frac{\pi}{4} = \frac{-125\sqrt{2}}{2}$

$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{12\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$ +4 P(x)

<-1> each error or missing term<-1> max for all extra terms, +..., misuse of equality

b. Find the coefficient of x^{22} in the Taylor series for f about x = 0.

$\frac{-5^{22}\sqrt{2}}{2(22!)}$ +1 magnitude, +1 sign

c. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$

$\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| \le \max\left|f^{(4)}(c)\right|\left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4 \le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$ +1 error bound in an appropriate inequality

d. Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about x = 0

$\int_0^x\left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right)dt = \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$ +2 third-degree Taylor polynoial for G about x = 0

<-1> each incorrect or missing term, <-1> max for all extra terms, +..., misuse of equality

2. ***The use of a calculator is NOT PERMITTED on this problem.*** The function f has a Taylor series about x = 2 that converges to f(x) for all x in the interval of convergence. The nth derivative of f at x = 2 is given by $f^{(n)}(2) = \frac{(n+1)!}{3^n}$ for all $n \geq 1$ and f(2) = 1.

a. Write the first four terms and the general term of the Taylor series for f about x = 2.

$f(2) = 1,\ f'(2) = \frac{2!}{3},\ f''(2) = \frac{3!}{3^2},\ f'''(2) = \frac{4!}{3^3}$ +1 coefficients $\frac{f^{(n)}(2)}{n!}$ in first four terms

$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \ldots + \frac{(n+1)!}{n!3^n}(x-2)^n + \ldots$ +1 powers of (x - 2) in first four terms

$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \ldots + \frac{n+1}{3^n}(x-2)^n + \ldots$ +1 general term

b. Find the radius of convergence for the Taylor series for f about x = 2. Show the work that leads to your answer.

$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2| = \frac{1}{3}|x-2| < 1 \text{ when } |x-2| < 3$

The radius of convergence is 3. +1 sets up ratio, +1 limit, +1 applies ratio test to conclude radius of convergence is 3

c. Let g be a function satisfying g(2) = 3 and g ‘ (x) = f(x) for all x. Write the first four terms and the general term of the Taylor series for g about x = 2.

$g(2) = 3,\ g'(2) = f(2),\ g''(2) = f'(2),\ g'''(2) = f''(2)$ +1 first four terms

$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \ldots + \frac{1}{3^n}(x-2)^{n+1} + \ldots$ +1 general term

d. Does the Taylor series for g defined in part (c) converge at x = -2? Give a reason for your answer.

No, the Taylor series does not converge at x = -2 because the geometric series only converges on the interval $|x - 2| < 3$ +1 answer with reason

3. ***The use of a calculator is REQUIRED on this problem.*** The function f has derivatives of all orders for all real numbers x. Assume f(2) = -3, f ' (2) = 5, f " (2) = 3, f "'(2) = -8.

a. Write the third-degree Taylor polynomial for f about x = 2 and use it to approximate f(1.5).

$T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$ +3 $T_3(f,2)(x)$ <-1> each error

$f(1.5) \approx T_3(f,2)(1.5) = -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{8}{6}(-0.5)^3 = -4.958$ +1 approximationg of f(1.5)

b. The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval [1.5, 2]. Use the Lagrange error bound on the approximation to f(1.5) found in part (a) to explain why $f(1.5) \ne -5$.

Lagrange Error Bound = $\frac{3}{4!}|1.5-2|^4 = 0.0078125$ +1 value of Lagrange Error Bound

$f(1.5) > -4.958 - 0.078125 = -4.966 > -5$ Therefore $f(1.5) \ne -5$ +1 explanation

c. Write the fourth-degree Taylor polynomial, P(x), for $g(x) = f(x^2 + 2)$ about x = 0. Use P to explain why g must have a relative minimum at x = 0.

$P(x) = T_4(g,0)(x) = T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$ +2 $T_4(g,0)(x)$

The coefficient of x in P(x) is g ' (0). This coefficient is 0, so g'(0) = 0. <-1> for each incorrect, missing or extra term

The coefficient of x^2 in P(x) is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at x = 0. +1 explanation

Overall: <-1> max for improper use of +... or equality

POW DUE 4/1/10

1. ***The use of a calculator is REQUIRED for this problem.*** The Taylor series about x = 5 for a certain function f converges to f(x) for all x in the interval of convergence. The nth derivative of f at x = 5 is given by

$f^{(n)}(5) = \frac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \frac{1}{2}$

a. Write the third-degree Taylor polynomial for f about x = 5.
b. Find the radius of convergence of the Taylor series for f about x = 5.
c. Show that the sixth-degree Taylor polynomial for f about x = 5 approximates f(6) with error less than 1/1000.

2. ***The use of a calculator is NOT PERMITTED for this problem.*** The function f is defined by the power series

$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2x+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + \frac{(-1)^n x^{2n}}{(2x+1)!} + \ldots$ for all real numbers x.

a. Find f ' (0) and f " (0). Determine whether f has a local maximum, a local minimum, or neither at x = 0. Give a reason for your answer.
b. Show that $1 - \frac{1}{3!}$ approximates f(1) with error less than 1/100.
c. Show that g = f(x) is a solution to the differential equation xy ' + y = cos x

3. ***The use of a calculator is NOT PERMITTED for this problem.*** A function f is defined by

$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \ldots + \frac{n+1}{3^{n+1}}x^n + \ldots$ for all x in the interval of convergence of the given power series.

a. Find the interval of convergence for this power series. Show the work that leads to your answer.
b. Find $\lim_{x\to 0} \frac{f(x) - \frac{1}{3}}{x}$
c. Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$
d. Find the sum of the series determined in part (c).