APCBC Problem of the Week Name ______________________________
DUE NO LATER THAN 4/16/10 at 8:00 AM

SOLUTIONS TO 4/1/10 POW

1. ***The use of a calculator is REQUIRED for this problem.*** The Taylor series about x = 5 for a certain function f converges to f(x) for all x in the interval of convergence. The nth derivative of f at x = 5 is given by

$f^{(n)}(5) = \frac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \frac{1}{2}$

a. Write the third-degree Taylor polynomial for f about x = 5.

$f'(5) = \frac{-1!}{2(3)}, f''(5) = \frac{2!}{4(4)}, f'''(5) = \frac{-3!}{8(5)}$ +3 $P_3(f,5)(x)$

$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$ <-1> each error or missing term

b. Find the radius of convergence of the Taylor series for f about x = 5.

$a_n = \frac{f^{(n)}(5)}{n!} = \frac{(-1)^n}{2^n (n+2)}$ +1 general term

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\frac{(-1)^n (x-5)^n}{2^n (n+2)}}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5| = \frac{|x-5|}{2} < 1$$

+1 sets up ratio test, +1 computes limit

The radius of convergence is 2 +1 applies ratio test to get radius of convergence

c. Show that the sixth-degree Taylor polynomial for f about x = 5 approximates f(6) with error less than 1/1000.

The Taylor series about x = 5 for the function f, when evaluated at x = 6, is an alternating series with absolute value of terms decreasing to 0. The error in approximating f(6) with the 6^{th} degree Taylor polynomial at x = 6 is less than the first omitted term in the series $|f(6) - P_6(f,5)(6)| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$ +1 error bound < 1/1000, +1 refers to an alternating series and indicates the error bound is found from the next term.

2. ***The use of a calculator is NOT PERMITTED for this problem.*** The function f is defined by the power series

$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2x+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + \frac{(-1)^n x^{2n}}{(2x+1)!} + \ldots$ for all real numbers x.

a. Find f ' (0) and f " (0). Determine whether f has a local maximum, a local minimum, or neither at x = 0. Give a reason for your answer.

$f'(0) = \text{coefficient of x term} = 0,\ f''(0) = 2(\text{coefficient of } x^2 \text{ term}) = 2\left(\frac{-1}{3!}\right) = \frac{-1}{3}$ +1 f '(0), +1 f "(0)

f has a local maximum at x = 0 because $f'(0) = 0$ and $f''(0) < 0$ +1 critical pt answer, +1 reason

b. Show that $1 - \frac{1}{3!}$ approximates f(1) with error less than 1/100.

$f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots + \frac{(-1)^n}{(2n+1)!} + \ldots$ This is an alternating series whose terms decrease in absolute value with limit 0.

Thus, the error is less than the first omitted term, so $\left|f(1) - \left(1 - \frac{1}{3!}\right)\right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$ +1 error bound < 1/100

c. Show that g = f(x) is a solution to the differential equation xy ' + y = cos x

$xy = xf(x) = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \ldots = \sin x \to xy' + y = (xy)' = (\sin x)' = \cos x$

+1 series for xf(x), +1 identifies series as sin x, +1 handles xy ' + y, +1 makes connection

3. ***The use of a calculator is NOT PERMITTED for this problem.*** A function f is defined by

$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \ldots + \frac{n+1}{3^{n+1}}x^n + \ldots$ for all x in the interval of convergence of the given power series.

a. Find the interval of convergence for this power series. Show the work that leads to your answer.

$$\lim_{n\to\infty}\left|\frac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x}{(n+1)3}\right| = \left|\frac{x}{3}\right| < 1$$

+1 sets up ratio test, +1 computes limit

At x = -3, the series is $\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{3}$ which diverges

+1 endpoint conclusion

At x = 3, the series is $\sum_{n=0}^{\infty}\frac{n+1}{3}$ which diverges

Therefore, the interval of convergence is $-3 < x < 3$.

+1 conclusion of ratio test

b. Find $\lim_{x\to 0}\frac{f(x)-\frac{1}{3}}{x}$ = $\lim_{x\to 0}\frac{f(x)-\frac{1}{3}}{x} = \lim_{x\to 0}\left(\frac{2}{3^2}+\frac{3}{3^3}x+\frac{4}{3^4}x^2+\ldots\right) = \frac{2}{9}$ +1 answer

c. Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$

$$\int_0^1 f(x)\,dx = \int_0^1\left(\frac{1}{3}+\frac{2}{3}x+\frac{3}{3^3}x^2+\ldots+\frac{n+1}{3^{n+1}}+\ldots\right)dx = \left(\frac{1}{3}x+\frac{1}{3^2}x^2+\frac{1}{3^3}x^3+\ldots+\frac{1}{3^{n+1}}x^{n+1}+\ldots\right)\Big|_{x=0}^{x=1}$$

$= \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots+\frac{1}{3^{n+1}}+\ldots$ +1 antidifferentiation of series, +1 first three terms for definite integral series, +1 general term

d. Find the sum of the series determined in part (c).

The series representing $\int_0^1 f(x)\,dx$ is a geometric series. Therefore $\int_0^1 f(x)\,dx = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$ +1 answer

POW DUE 4/16/10

The use of a calculator is REQUIRED on these problems.

1. An object moving along a curve in the xy-plane has position (x(t), y(t)) at time $t \geq 0$ with $\frac{dx}{dt} = 12t - 3t^2$ and $\frac{dy}{dt} = \ln\left(1 + (t-4)^4\right)$. At time t = 0, the object is at position (-13, 5). At time t = 2, the object is at point P with x-coordinate 3.
   a. Find the acceleration vector at time t = 2 and the speed at time t = 2.
   b. Find the y-coordinate of P.
   c. Write an equation for the line tangent to the curve at P.
   d. For what value of t, if any, is the object at rest? Explain your reasoning.

2. A water tank at Camp Newton holds 1200 gallons of water at time t = 0. During the time interval $0 \leq t \leq 18$ hours, water is pumped into the tank at the rate $W(t) = 95\sqrt{t}\sin^2\frac{t}{6}$ gallons per hour. During the same time interval, water is removed from the tank at the rate $R(t) = 275\sin^2\frac{t}{3}$ gallons per hour.
   a. Is the amount of water in the tank increasing at time t = 15? Why or why not?
   b. To the nearest whole number, how many gallons of water are in the tank at time t = 18?
   c. At what time t, for $0 \leq t \leq 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.
   d. For t > 18, no water is pumped into the tank, but water continues to be removed at the rate R(t) until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

3. The Taylor series about x = 0 for a certain function f converges to f(x) for all x in the interval of convergence. The nth derivative of f at x = 0 is given by $f^n(0) = \frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2}$ for $n \geq 2$. The graph of f has a horizontal tangent line at x = 0, and f(0) = 6.
   a. Determine whether f has a relative maximum, a relative minimum, or neither at x = 0. Justify your answer.
   b. Write the third degree Taylor polynomial for f about x = 0.
   c. Find the radius of convergence of the Taylor series for f about x = 0. Show the work that leads to your answer.